# Weak* topology and its application

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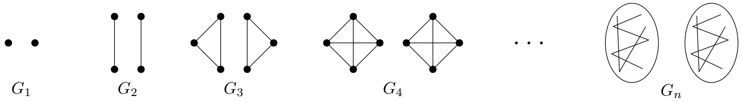
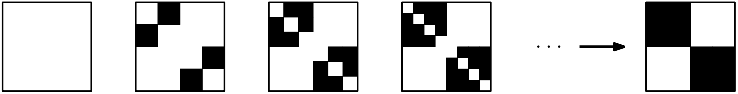
March 30, 2018

# Graphons and cut distance convergence




Represent graphs as adjacency matrices:



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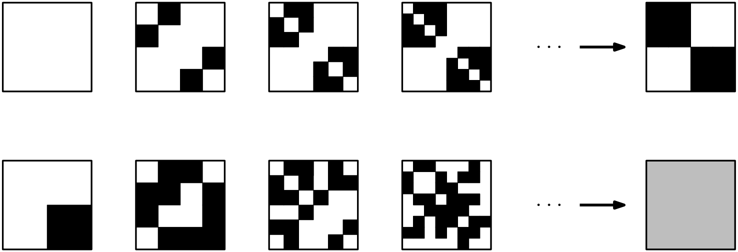

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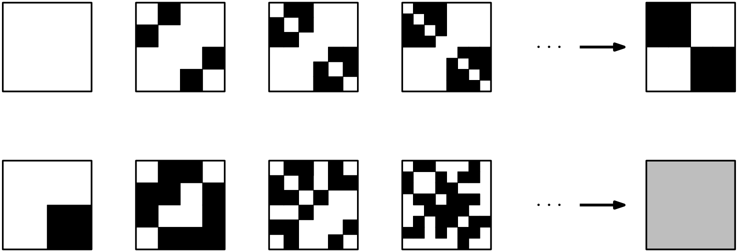
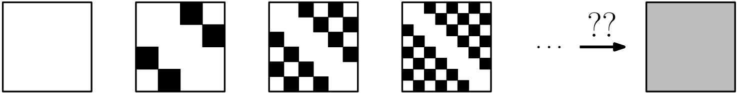


But one has to be careful:



# Cut distance topology

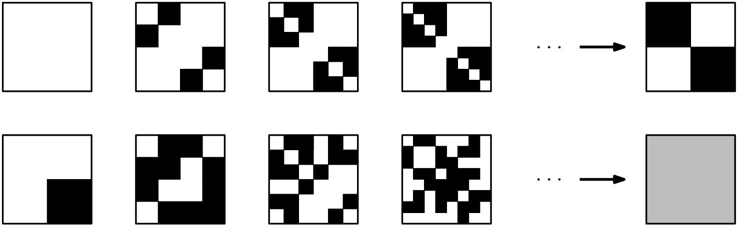
1) Compare the number of edges inside any vertex set:

$$d_\square(U,V) = \sup_{S,T\subseteq[0,1]} \left| \int_{S\times T} U(x,y) - V(x,y) \right| .$$

2) Minimise over permutations of the adjacency matrix:

$$\delta_\square(U,V) = \inf_\pi d_\square(U,V^\pi) .$$

where $\pi : [0,1] \to [0,1]$ runs over all measure preserving bijections and $U^\pi(x,y) = U(\pi(x),\pi(y))$.



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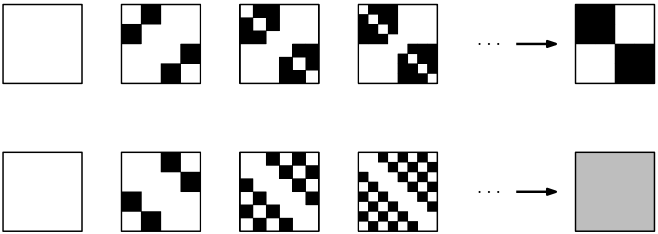
**Lovász-Szegedy'06:** For every sequence $U_1, U_2, \ldots$ there exist $\pi_{n_1}, \pi_{n_2}, \ldots$ and $V$ such that $U_{n_1}^{\pi_{n_1}}, U_{n_2}^{\pi_{n_2}}, \ldots \xrightarrow{d_{\Box}} V$.

# Weak* convergence

$$U_1, U_2, \ldots \xrightarrow{w^*} V \iff \forall S, T \subseteq [0,1] : \lim_{n\to\infty} \int_{S\times T} U_n = \int_{S\times T} V \,.$$



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**Observation:** For every sequence $U_1, U_2, \ldots$ there exist $\pi_{n_1}, \pi_{n_2}, \ldots$ and $V$ such that $U_{n_1}^{\pi_{n_1}}, U_{n_2}^{\pi_{n_2}}, \ldots \xrightarrow{w^*} V$.

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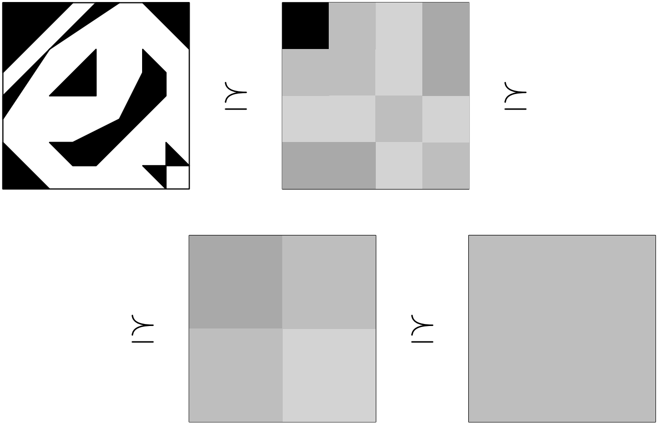

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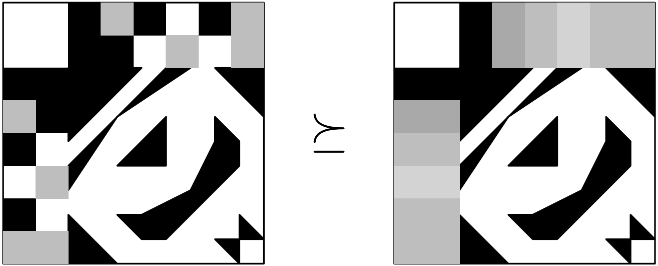

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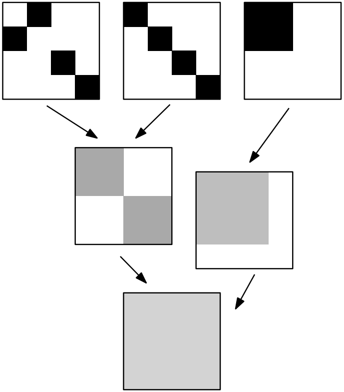
$$\begin{array}{|c|c|}\hline U & 0 \\ \hline 0 & V \\ \hline \end{array} \quad \succeq \quad \begin{array}{|c|}\hline \frac{U+V}{4} \\ \hline \end{array}$$

# Weak* convergence: structuredness order



- The relation $\succeq$ is a preorder.
- $U \succeq V$ and $V \succeq U \iff \delta_\square(U, V) = 0$
- Maximal elements are zero-one graphons, minimal are constant graphons.

# Weak* convergence: compatible parameters

What functions $\Theta : \mathcal{W}_0 \to \mathbb{R}$ are compatible with the structuredness order, i.e., $U \succeq V$ implies $\Theta(U) \geq \Theta(V)$?

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Suppose that $\Theta$ satisfies:

- $\Theta$ is continuous in $L_1$,
- $\Theta(U) = \Theta(U^\pi)$ for measure preserving bijection $\pi$,
- $\frac{1}{2}\Theta(U) + \frac{1}{2}\Theta(V) \geq \Theta\left(\frac{U+V}{2}\right)$.

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1) It suffices to show that the value of $\Theta$ does not increase after averaging any graphon on any partition, i.e., $\Theta(U^{\bowtie\mathcal{P}}) \leq \Theta(U)$.
2) Approximate $U^{\bowtie\mathcal{P}}$ by versions of $U$, i.e.,
$U^{\bowtie\mathcal{P}} \overset{L_1}{\approx} \frac{1}{n}(U^{\pi_1} + \cdots + U^{\pi_n})$ and use convexity.

# Weak* convergence: compatible parameters

Note that the parameter $t(H, \cdot)$ is both continuous in $L_1$ and $t(H, U) = t(H, V)$ if $\delta_\square(U, V) = 0$.

$$t(H, U) = \int_{[0,1]^{|V(H)|}} \prod_{ij \in E(H)} U(x_i, x_j)$$

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A graph $H$ is called weakly norming, if $t(H,\cdot)^{1/|E(H)|}$ is convex.

**Theorem (Hatami'10)**

Hypercubes, complete bipartite graphs, even cycles,... are weakly norming, thus compatible. Nonbipartite graphs, nonstar trees,... are not weakly norming.

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A graph $H$ is called Sidorenko, if $t(H,\cdot)$ is minimised by constant graphons.
Each weakly norming graph is compatible with structuredness order and thus Sidorenko.

# Weak* convergence: compatible parameters

**Theorem (Kráľ, Martins, Pach, Wrochna'18+)**

There are edge-transitive graphs that are not compatible, thus not weakly norming

**Question (Kráľ, Martins, Pach, Wrochna'18+)**

Is it true that every connected graph $H$ is weakly norming if and only if it is compatible with structuredness order?

# Weak* convergence: compatible parameters

Idea of proof: for connected $H$ compute its homomorphism density in these two graphons.

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Recover the constant loss via tensor power trick.