Embedding trees in dense graphs

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joint work with T. Klimošová and D. Piguet

Extremal graph theory

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Alternative definition: substructures in graphs

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Graph G has n vertices. If G has more than $n^2/4$ edges then it contains a triangle.

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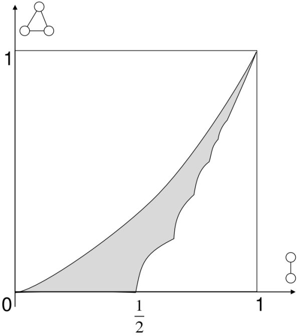
Density of edges vs. density of triangles (Razborov 2008)



(image from the book of Lovász)

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The answer for C_4 is of order $n^{3/2}$, lower bound via finite projective planes.
What is the answer for trees?

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Partial results:

- special trees (paths – Erdős, Gallai 1959)
- special graphs (without C_4 – Saclé, Wozniak 1997)
- n and k differ by constant (Görlich, Żak 2016)

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One can get reasonably close if the size of the tree is comparable with the size of the graph. Below Δ is maximum degree and $\overline{\deg}$ average degree.

Theorem (R. 2019), also (Besomi, Pavez-Signé, Stein 2019+)

Let $\mathcal{T}$ be a class of trees such that $\forall T \in \mathcal{T} : \Delta(T) \in o(|T|)$.
Then any graph G with $\overline{\deg}(G) = |T| + o(|G|)$ contains any $T \in \mathcal{T}$.

Loebl-Komlós-Sós

Conjecture (Loebl, Komlós, Sós 1995)

If at least $n/2$ vertices of G have degree at least k, then G contains any tree with $k+1$ vertices as a subgraph.

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If at least rn vertices of G have degree at least k, then G contains any tree with $k+1$ vertices and at most $r(k+1)$ vertices in one colour class as a subgraph.

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Theorem (Klimošová, Piguet, R. 2019+)

If at least rn vertices of G have degree at least $k+o(n)$, then G contains any tree with $k+1$ vertices and at most $r(k+1)$ vertices in one colour class as a subgraph.

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Turns out that one can get this 'r' trade-off also in the proof of previous Erdős-Sós result.

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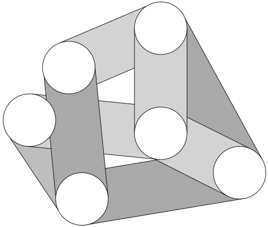

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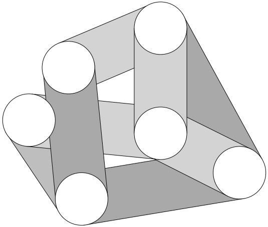

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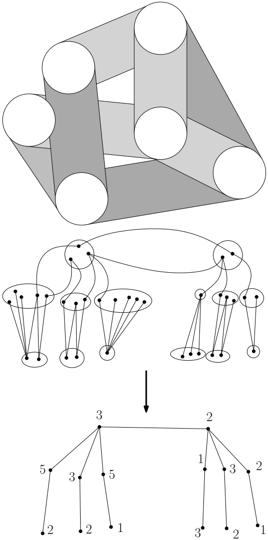

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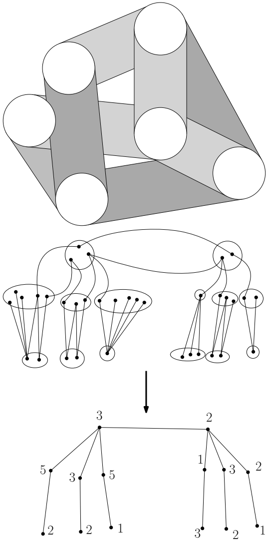
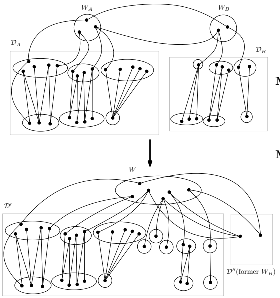
- We reduced the problem to a certain fractional variant of itself. But now we have much simpler tree structure to work with.

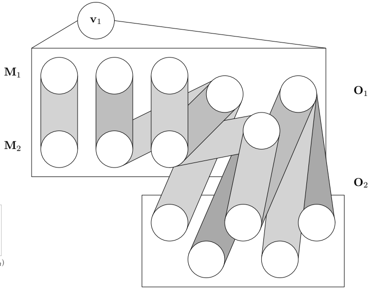


Proof of the Erdős-Sós result

Proof.

Condition on the maximum degree actually gives even simpler decomposition. After decomposition of G and T look at a high degree cluster of G and a maximal matching in its neighbourhood. Provide (almost) greedy algorithm for embedding. □





Proof of the Loebl-Komlós-Sós result

Proof.

After decomposition of G and T 'discharge' into several configurations, embedding for each one being straightforward. □