A Non-Column Tile in $\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ Glued from Two Slopes with Periodic Supports Neither of Which Tiles $\mathbb{Z}$

Abstract

We work in $G = \mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with counting measure on $\mathbb{Z}$ and normalized Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. We answer the following question affirmatively: there exists a tile $A \subset G$ which is not a column and a tiling $T \subset G$ such that $\mathbb{Z}$ decomposes as a disjoint union $\mathbb{Z} = X \sqcup Y$ of periodic sets, $T = T_1 \sqcup T_2$, with

$$A + T_1 = X \times \mathbb{R}/\mathbb{Z}, \qquad A + T_2 = Y \times \mathbb{R}/\mathbb{Z},$$

where T_1 is $(p, 0)$-periodic for some $p \in \mathbb{Z}_{\geq 2}$, T_2 is $(p, p\beta)$-periodic for some irrational $\beta \in [0, 1)$, and neither X nor Y tiles $\mathbb{Z}$ by finitely many disjoint translates. The construction uses a three-column tile and two single-coset tilings at slopes $\alpha = 0$ and $\alpha = \beta$ glued to prescribed columns. All equalities and disjointness statements are understood almost everywhere (a.e.) with respect to the product measure; we use half-open arcs to avoid boundary issues.

1 Setup and the Question

Group and measure. Let $G = \mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with addition $(m, \theta) + (n, \varphi) = (m+n, \theta+\varphi \bmod 1)$. On $\mathbb{Z}$ we use counting measure and on $\mathbb{R}/\mathbb{Z}$ normalized Lebesgue measure μ. All unions/coverings/disjointness are meant a.e.

Tiles and tilings. A *tile* is a finite union of column slices

$$A = \bigcup_{i=1}^{\ell} \{n_i\} \times I_i, \qquad n_i \in \mathbb{Z},\ I_i \subset \mathbb{R}/\mathbb{Z} \text{ half-open intervals.}$$

For $\alpha \in [0, 1)$ and $c \in \mathbb{R}/\mathbb{Z}$ we write

$$T_{\alpha,c} := \{(m, m\alpha + c) : m \in \mathbb{Z}\} \subset G.$$

Given $T \subset G$, we say that A *tiles* G *with* T if $A + T = G$ a.e. and $(A + t) \cap (A + t')$ has measure zero for all distinct $t, t' \in T$.

Columns. We call A a *column* if there exist a finite $C \subset \mathbb{Z}$ and a finite $\Lambda \subset \mathbb{R}/\mathbb{Z}$ such that $A \subset C \times \mathbb{R}/\mathbb{Z}$ and

$$\bigcup_{\lambda \in \Lambda} \big(A + (0, \lambda)\big) = C \times \mathbb{R}/\mathbb{Z}, \qquad (A + (0, \lambda)) \cap (A + (0, \lambda')) \text{ has measure } 0 \text{ for } \lambda \neq \lambda'. \tag{1}$$

Question. *Does there exist a tile A and a tiling T for which we can write $\mathbb{Z} = X \sqcup Y$ (disjoint union), $T = T_1 \sqcup T_2$, such that X, Y are periodic,*

- *$A + T_1 = X \times \mathbb{R}/\mathbb{Z}$ and $A + T_2 = Y \times \mathbb{R}/\mathbb{Z}$;*
- *T_1 is $(\ell_1, 0)$-periodic and T_2 is (ℓ_2, α')-periodic for some irrational α';*
- *A is not a column;*
- *and moreover neither X nor Y tiles $\mathbb{Z}$ by finitely many disjoint translates?*

2 Single-Coset Criterion and Gluing to Prescribed Columns

Lemma 1 (Fiber computation and single-coset criterion)**.** *Let $A = \bigcup_{i=1}^{\ell}\{n_i\} \times I_i$ with half-open $I_i \subset \mathbb{R}/\mathbb{Z}$ and define*

$$S(\alpha) := \bigcup_{i=1}^{\ell}(I_i - n_i\alpha) \subset \mathbb{R}/\mathbb{Z}.$$

Then the fiber over $r \in \mathbb{Z}$ of $A + T_{\alpha,c}$ equals $c + r\alpha + S(\alpha)$. In particular, $A + T_{\alpha,c}$ tiles G a.e. (for some/every c) iff the translates $\{I_i - n_i\alpha\}$ are pairwise a.e. disjoint and $S(\alpha) = \mathbb{R}/\mathbb{Z}$ a.e.

Proof. Points $(n_i, \theta) \in A$ contribute to the r-th fiber after adding $(r - n_i, (r - n_i)\alpha + c)$, giving $\theta + (r - n_i)\alpha + c$. Thus the fiber is $c + r\alpha + \bigcup_i(I_i - n_i\alpha)$. Disjointness/coverage in a fixed fiber reduce to the stated conditions; different r give disjoint first coordinates. □

Lemma 2 (Restricted-coset gluing)**.** *Under the hypothesis of Lemma 1 for a fixed α, let $X \subset \mathbb{Z}$ and $c \in \mathbb{R}/\mathbb{Z}$. Define*

$$T_X(\alpha, c) := \bigcup_{i=1}^{\ell}\{(m, m\alpha + c) : m \in X - n_i\}.$$

Then $A + T_X(\alpha, c) = X \times \mathbb{R}/\mathbb{Z}$ a.e., with fiberwise a.e. disjointness. If X is p-periodic, then $T_X(\alpha, c)$ is $(p, p\alpha)$-periodic.

Proof. Fix $r \in \mathbb{Z}$. A translate from $T_X(\alpha, c)$ contributes to the r-th fiber iff $r = n_i + m$ for some i with $m \in X - n_i$, i.e. iff $r \in X$. If $r \notin X$, the fiber is empty; if $r \in X$ it equals $c + r\alpha + S(\alpha) = \mathbb{R}/\mathbb{Z}$ a.e., and disjointness holds by Lemma 1. Periodicity is immediate from $m \in X - n_i \Rightarrow m + p \in X - n_i$. □

Lemma 3 (Column obstruction)**.** *Suppose $A \subset C \times \mathbb{R}/\mathbb{Z}$ with C finite and there exists finite $\Lambda \subset \mathbb{R}/\mathbb{Z}$ such that (1) holds with pairwise a.e. disjoint translates. Let $A_k := \{\theta : (k, \theta) \in A\}$. Then $\mu(A_k) = 1/|\Lambda|$ for all $k \in C$. In particular, if two nonempty slices A_k have different measures, A is not a column.*

Proof. For a.e. θ, $\sum_{\lambda\in\Lambda} \mathbf{1}_{A_k}(\theta - \lambda) = 1$. Integrate and use translation invariance. □

3 Arithmetic Obstructions to Tiling $\mathbb{Z}$ by Finitely Many Translates

Lemma 4 (Density obstruction)**.** *Let $E \subset \mathbb{Z}$ and $F \subset \mathbb{Z}$ be finite such that $\{E + f : f \in F\}$ are pairwise disjoint and $\bigcup_{f\in F}(E + f) = \mathbb{Z}$. Then the natural density $d(E)$ exists and equals $1/|F|$.*

Proof. For $N \in \mathbb{Z}_{\geq 0}$,

$$2N+1 = \sum_{f \in F} |(E+f) \cap [-N,N]|.$$

Divide by $2N+1$. For each fixed f, the difference between the normalized counts of E and $E+f$ tends to zero as $N \to \infty$, hence $\sum_{f\in F} d_N(E+f) \to |F| \cdot \lim d_N(E) = 1$. □

Lemma 5 (Finite cyclic obstruction for periodic sets)**.** *Let $q \geq 2$ and $E \subset \mathbb{Z}$ be q-periodic with residue set $R \subset \mathbb{Z}/q\mathbb{Z}$ of size $a = |R|$. If there exists finite $F \subset \mathbb{Z}$ such that $\{E+f : f \in F\}$ are pairwise disjoint and $\bigcup_{f\in F}(E+f) = \mathbb{Z}$, then the residues $F \pmod q$ are all distinct and $a \cdot |F| = q$. In particular $a \mid q$.*

Proof. Reduce the tiling to $\mathbb{Z}/q\mathbb{Z}$. For each $r \in \mathbb{Z}/q\mathbb{Z}$ there is a unique pair $(e,f) \in R \times (F \bmod q)$ with $e+f \equiv r$, whence $q = |R| \cdot |F \pmod q)|$. For fixed e, $f \mapsto e+f$ is injective modulo q, so the residues of F are distinct and $|F \pmod q)| = |F|$, giving $a|F| = q$. □

Corollary 1 (Prime case and complements)**.** *If $q = p$ is prime and $E \subset \mathbb{Z}$ is p-periodic with $|R| = a$, then E tiles $\mathbb{Z}$ by finitely many disjoint translates iff $a \in \{1, p\}$. In particular, for $1 < a < p-1$, neither E nor its complement tiles $\mathbb{Z}$.*

Proof. Apply Lemma 5 to E and to its complement (with residue size $p - a$). □

4 Periodicity of Coset-Slope Sets and Disjointization

Lemma 6 (Periodicity vector)**.** *Let $T = \{(m, m\alpha + c) : m \in M\} \subset G$. If T is invariant under addition of (ℓ, γ), then M is ℓ-periodic and $\gamma \equiv \ell\alpha \pmod 1$. Conversely, if these hold then $T + (\ell, \gamma) = T$.*

Proof. Invariance gives $(m+\ell, (m+\ell)\alpha + c + \gamma) = (m', m'\alpha + c) \in T$ for some $m' \in M$. Comparing first coordinates forces $m' = m + \ell$ and then $(m+\ell)\alpha + c + \gamma \equiv m\alpha + c$, i.e. $\gamma \equiv -\ell\alpha \equiv \ell\alpha \pmod 1$. The converse is immediate. □

Lemma 7 (Disjointizing the constants)**.** *Let β be irrational, $X, Y \subset \mathbb{Z}$, and define*

$$T_1 := \bigcup_{i=1}^{\ell} \{(m, c_1) : m \in X - n_i\}, \qquad T_2 := \bigcup_{i=1}^{\ell} \{(m, m\beta + c_2) : m \in Y - n_i\}.$$

There exists $c_1 \in \mathbb{R}/\mathbb{Z}$ such that $T_1 \cap T_2 = \emptyset$.

Proof. If $(m, c_1) \in T_1 \cap T_2$ then $m \in \big(\cup_i (X - n_i)\big) \cap \big(\cup_i (Y - n_i)\big)$ and $c_1 \equiv m\beta + c_2 \pmod 1$. The set of forbidden c_1 is countable, hence avoidable. □

5 A Three-Column Tile for Two Slopes and Non-Columnness

Proposition 1 (Three-column single-coset tile for $\alpha = 0$ and $\alpha = \beta$)**.** *Let $\beta \in [0,1)$ be irrational. Then there exists a three-column tile*

$$A = \{n_1\} \times I_1 \ \cup \ \{n_2\} \times I_2 \ \cup \ \{n_3\} \times I_3$$

such that the single-coset criterion (Lemma 1) holds both for $\alpha = 0$ and for $\alpha = \beta$. Moreover, the slice measures $\mu(I_1), \mu(I_2), \mu(I_3)$ are not all equal, hence A is not a column by Lemma 3.

Proof. Choose distinct $t_1, t_2, t_3 \in \mathbb{Z}$ with distinct fractional parts $b_i = \{t_i\beta\}$, relabeled so that $0 \le b_1 < b_2 < b_3 < 1$. Set lengths $L_1 := b_2 - b_1$, $L_2 := b_3 - b_2$, $L_3 := 1 - (b_3 - b_1)$ (positive and summing to 1). Let $n_1 := 0$, $n_2 := t_1 - t_3$, $n_3 := t_2 - t_3$ and define $a_i := b_i + n_i\beta \pmod 1$, $I_i := [a_i, a_i + L_i)$.

For $\alpha = \beta$ one has $I_i - n_i\beta = [b_i, b_i + L_i)$, the three consecutive arcs $[b_1, b_2)$, $[b_2, b_3)$, $[b_3, b_1)$, which partition $\mathbb{R}/\mathbb{Z}$ a.e.

For $\alpha = 0$, write $t_i\beta = q_i + b_i$ with $q_i \in \mathbb{Z}$. Then modulo 1,

$$a_1 \equiv b_1, \qquad a_3 \equiv b_2, \qquad a_2 \equiv b_1 - (b_3 - b_2).$$

It follows that the cyclic gaps between a_1, a_3, a_2 are L_1, L_3, L_2, hence I_1, I_2, I_3 are pairwise disjoint with total length 1, i.e. they cover $\mathbb{R}/\mathbb{Z}$ a.e.

Finally, $L_1 = \{(t_2 - t_1)\beta\}$ and $L_2 = \{(t_3 - t_2)\beta\}$ are irrational; if $L_1 = L_2$ then $L_3 = 1 - 2L_1 \ne L_1$ (else $L_1 = 1/3$ rational). Thus the slice measures are not all equal, and Lemma 3 implies A is not a column. $\square$

6 Main Theorem: Gluing on Complementary Periodic Supports that Do Not Tile

Theorem 1 (Affirmative answer with non-tiling X, Y)**.** *Let $\beta \in [0, 1)$ be irrational and let p be prime. Choose a residue set $R \subset \mathbb{Z}/p\mathbb{Z}$ with $1 < |R| < p - 1$ and define*

$$X := \{n \in \mathbb{Z} : \ n \bmod p \in R\}, \qquad Y := \mathbb{Z} \setminus X.$$

Then there exist a tile $A \subset G$ which is not a column and disjoint translation sets $T_1, T_2 \subset G$ such that, writing $T := T_1 \sqcup T_2$,

(i) $A + T_1 = X \times \mathbb{R}/\mathbb{Z}$ and $A + T_2 = Y \times \mathbb{R}/\mathbb{Z}$ a.e., hence $A + T = \mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ a.e.;

(ii) T_1 is $(p, 0)$-periodic and T_2 is $(p, p\beta)$-periodic;

(iii) neither X nor Y tiles $\mathbb{Z}$ by finitely many disjoint translates.

Proof. Let A be the three-column tile from Proposition 1. Apply Lemma 2 with $\alpha = 0$ and the p-periodic set X to obtain $T_1 := T_X(0, c_1)$ satisfying $A + T_1 = X \times \mathbb{R}/\mathbb{Z}$ a.e.; similarly, apply Lemma 2 with $\alpha = \beta$ and Y to obtain $T_2 := T_Y(\beta, c_2)$ satisfying $A + T_2 = Y \times \mathbb{R}/\mathbb{Z}$ a.e. Since $X \sqcup Y = \mathbb{Z}$, the two coverings are over disjoint columns, hence $A + (T_1 \cup T_2) = \mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ a.e. By Lemma 7, we may choose c_1 so that $T_1 \cap T_2 = \emptyset$, i.e. $T = T_1 \sqcup T_2$.

Periodicity follows from Lemma 6: X and Y are p-periodic, so T_1 is $(p, 0)$-periodic and T_2 is $(p, p\beta)$-periodic. Finally, by Corollary 1 (applied to $|R|$ and to $p - |R|$), neither X nor Y tiles $\mathbb{Z}$ by finitely many disjoint translates. $\square$

Remark 1 (Two columns are impossible for dual slopes)**.** *Let $A = \{n_1\} \times I_1 \ \cup \ \{n_2\} \times I_2$ with half-open intervals $I_1, I_2 \subset \mathbb{R}/\mathbb{Z}$. If $\{I_1, I_2\}$ is a.e. disjoint with union $\mathbb{R}/\mathbb{Z}$ and, for some irrational β, also $\{I_1 - n_1\beta,\ I_2 - n_2\beta\}$ is a.e. disjoint with union $\mathbb{R}/\mathbb{Z}$, then necessarily $n_1 = n_2$. In particular, a two-column tile cannot simultaneously satisfy the single-coset criterion for $\alpha = 0$ and $\alpha = \beta$.*

Proof sketch. From $\alpha = 0$ we have $I_2 = \mathbb{R}/\mathbb{Z} \setminus I_1$ a.e. From $\alpha = \beta$, $I_2 - n_2\beta = \mathbb{R}/\mathbb{Z} \setminus (I_1 - n_1\beta)$ a.e., hence $I_1 = I_1 - (n_1 - n_2)\beta$ a.e., which forces $(n_1 - n_2)\beta \equiv 0 \pmod 1$. Since β is irrational, $n_1 = n_2$. $\square$

Remark 2 (Explicit instance)**.** *Take $\beta = \sqrt{2}/12$, $p = 5$, and $R = \{0, 1\}$. Then $X = \{n \equiv 0, 1 \pmod 5\}$ and $Y = \mathbb{Z} \setminus X$. Choose $t_1 = 0$, $t_2 = 1$, $t_3 = 3$; then $b_1 = 0$, $b_2 = \beta$, $b_3 = 3\beta$, giving lengths $L_1 = \beta$, $L_2 = 2\beta$, $L_3 = 1 - 3\beta$, and columns $n_1 = 0$, $n_2 = -3$, $n_3 = -2$. Proposition 1 and Theorem 1 apply verbatim.*

Conclusion

We constructed a non-column tile A admitting two single-coset tilings at slopes 0 and an irrational β, glued onto complementary periodic supports X and Y of common period p, such that neither X nor Y tiles $\mathbb{Z}$ by finitely many translates. This gives an affirmative answer to the stated question and strictly strengthens the example obtainable by "gluing in $2\mathbb{Z}$" from the original paper.