A Sharp Linear Lower Bound in the Multi-Function Partition Problem

Abstract

We consider $k \geq 2$ functions $f_1, \dots, f_k : E \to F$ with pointwise distinct outputs and the associated *conflict graph* G on E, where xy is an edge iff there exist $p \neq q$ with $f_p(x) = f_q(y)$. Let

$$n^* \;:=\; \max_{z \in F} \min_{t \in [k]} |f_t^{-1}(z)|$$

be the intrinsic fibre parameter. We prove that any general linear upper bound of the form $\chi(G) \leq c(k)\, n^*$ (valid for all such instances with k functions) must satisfy $c(k) \geq k$ for every $k \geq 2$. The proof uses a permutation construction with $\chi(G) = k!$ and $n^* = (k-1)!$.

1 Setup

Let $k \geq 2$ and $f_1, \dots, f_k : E \to F$ satisfy *pointwise distinctness*:

$$\forall x \in E, \quad \text{the values } f_1(x), \dots, f_k(x) \text{ are pairwise distinct.}$$

Define the *conflict graph* G on vertex set E by joining distinct $x, y \in E$ whenever

$$\exists\, p \neq q \in [k] \;\text{ such that }\; f_p(x) = f_q(y). \tag{1}$$

A partition $E = \bigsqcup_i E_i$ has the property that $f_p(E_i) \cap f_q(E_i) = \varnothing$ for all $p \neq q$ if and only if each E_i is a stable set in G; hence the minimum number of parts equals $\chi(G)$.

We measure the "size" of fibres by

$$n^* \;:=\; \max_{z \in F} \min_{t \in [k]} |f_t^{-1}(z)|, \tag{2}$$

the smallest integer n for which the hypothesis "$\forall z\ \exists t$ with $|f_t^{-1}(z)| \leq n$" holds.

2 A permutation construction

Let $E = S_k$ be the set of all permutations of $[k] = \{1, \dots, k\}$, take $F = [k]$, and define

$$f_i(\pi) \;:=\; \pi(i) \qquad (\pi \in S_k,\ i \in [k]). \tag{3}$$

Pointwise distinctness holds since $\{\pi(1), \dots, \pi(k)\} = [k]$ for each π.

Lemma 1. *For the instance* (3)*, the conflict graph G is complete on $|E| = k!$ vertices.*

Proof. Let $\pi \neq \sigma \in S_k$. Choose a symbol $s \in [k]$ such that $\pi^{-1}(s) \neq \sigma^{-1}(s)$ (some symbol moves position). Then with $p := \pi^{-1}(s)$ and $q := \sigma^{-1}(s)$ we have $p \neq q$ and

$$f_p(\pi) = \pi(p) = s = \sigma(q) = f_q(\sigma).$$

By (1), π and σ are adjacent. Hence every pair is adjacent and G is complete. $\square$

Lemma 2. *For the instance* (3)*, for every $z \in [k]$ and $i \in [k]$ the fibre size $|f_i^{-1}(z)| = (k-1)!$. Consequently $n^* = (k-1)!$.*

Proof. Fix i and z. The constraint $f_i(\pi) = z$ is $\pi(i) = z$. The remaining $k-1$ positions can be permuted arbitrarily, giving $(k-1)!$ permutations. For fixed z, the minimum of $|f_i^{-1}(z)|$ over i is therefore $(k-1)!$, and taking the maximum over z yields $n^* = (k-1)!$. $\square$

Combining Lemmas 1 and 2 we obtain

$$\chi(G) \;=\; k! \;=\; k \cdot (k-1)! \;=\; k\, n^*. \tag{4}$$

3 Main lower bound

Theorem 3 (Sharp linear lower bound)**.** *Let $c : \{2, 3, \dots\} \to \mathbb{R}_{\geq 0}$ be any function with the property that* for every *instance of k functions as above, the associated conflict graph G satisfies*

$$\chi(G) \;\leq\; c(k)\, n^*.$$

Then necessarily $c(k) \geq k$ for every $k \geq 2$.

Proof. Apply the assumed bound to the permutation instance (3). By (4) we have $\chi(G) = k\, n^*$, hence

$$k\, n^* \;=\; \chi(G) \;\leq\; c(k)\, n^*.$$

Since $n^* > 0$, it follows that $c(k) \geq k$. $\square$

Remark 4. The proof is purely extremal and shows that the linear coefficient in any universal bound of the form $\chi(G) \leq c(k)\, n^*$ cannot be smaller than k. In particular, no bound independent of k is possible, and any attempt to use a constant $< k$ fails already on the permutation family.

Acknowledgement. The permutation construction is folklore in related coloring problems.