A skew version of the Loebl–Komlós–Sós conjecture

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Abstract

Loebl, Komlós, and Sós conjectured that any graph such that at least half of its vertices have degree at least k contains every tree of order at most $k+1$. We propose a skew version of this conjecture. We consider the class of trees of order at most $k+1$ of given *skew*, that is, such that the sizes of the colour classes of the trees have a given ratio. We show that our conjecture is asymptotically correct for dense graphs. The proof relies on the regularity method. Our result implies bounds on Ramsey number of several trees of given skew.

Keywords: extremal graph theory, trees, Loebl–Komlós–Sós Conjecture, regularity lemma

1 Introduction and results

Many problems in extremal graph theory ask whether a certain density condition imposed on a host graph forces the containment of a given subgraph H. Typically, the density condition is expressed by the average or minimum degree. For example, the Erdős-Stone Theorem [3] essentially determines the average degree condition guaranteeing the containment of a fixed non-bipartite graph H. However, for a general bipartite graph H the problem is wide open. One of the most notorious problems in this direction is the Erdős-Sós conjecture from 1962, which determines the average degree forcing a copy of each tree T of a given size k. A solution of this conjecture for large k has been announced in the early 1990's by Ajtai, Komlós, Simonovits, and Szemerédi [1]. A trivial bound for the average degree guaranteeing containment of T is $2k$. Indeed, we can find a subgraph of minimum degree at least k and then embed T using the greedy procedure. A different approach to the problem is to relax the condition of minimum degree by investigating how many vertices of degree k guarantee the containment of a tree of order $k+1$. The Loebl-Komlós-Sós conjecture asserts that only half of the vertices need to have degree at least k. The conjecture has been solved exactly for large dense graphs [2,9] and proved to be asymptotically true for sparse graphs [4,5,6,7]. The Loebl-Komlós-Sós conjecture is best possible when we consider the class of all trees of order $k+1$, in particular, it is tight for paths. To observe this, consider a graph consisting of a disjoint union of copies of a graph H of order $k+1$ consisting of a clique of size $\lfloor \frac{k+1}{2} \rfloor - 1$, an independent set on the remaining vertices, and the complete bipartite graph between the two sets. Almost half of the vertices of this graph have degree k, but it does not contain a path on $k+1$ vertices as a subgraph.

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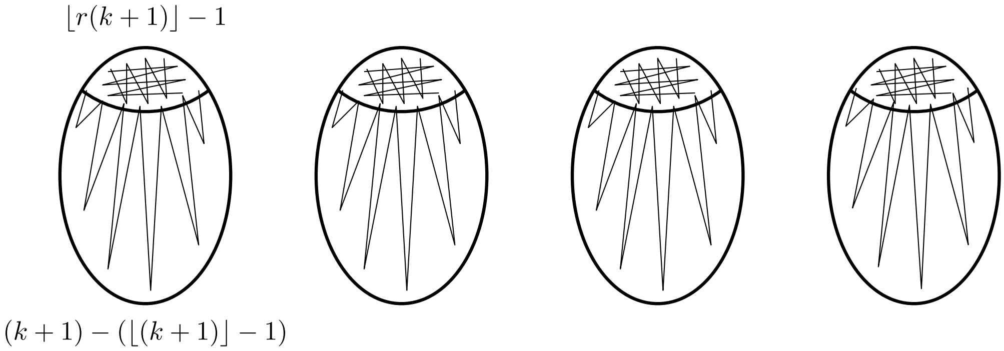

Fig. 1. The graph showing the tightness of Conjecture 1.1 is a disjoint union of graphs of order $k+1$.

A natural question is whether fewer vertices of degree k suffice when one considers only a restricted class of trees. Specifically, Simonovits asked [personal communication], whether it is the case for trees of given *skew*, that is, the ratio of sizes of the smaller and the larger colour classes is bounded by a constant smaller than 1. We propose the following conjecture.

Conjecture 1.1 *Any graph of order n with at least rn vertices of degree at least k contains every tree of order at most $k+1$ with colour classes V_1, V_2 such that $|V_1| \le r \cdot (k+1)$.*

We have verified that the conjecture holds both for trees of diameter at most five and for paths on at most $2r(k+1)$ vertices.

If true, the conjecture is best possible. Indeed, given $r \in (0, 1/2]$, consider a graph consisting of a disjoint union of copies of a graph H with $k+1$ vertices consisting of a clique of size $\lfloor r(k+1)\rfloor - 1$, an independent set on the remaining vertices and the complete bipartite graph between the two sets (see Figure 1). Such a graph does not contain a path on $2\lfloor r(k+1)\rfloor$ vertices.

Considering the structure of the above mentioned graph witnessing the tightness of our conjecture, it might seem feasible to strengthen the conjecture by replacing the condition on the size of the smaller colour class by the same condition on the size of the complement of a maximal independent set. However, this is not possible; a complete bipartite graph $K_{(k-1)/2,k}$ does not contain a bistar $B_{(k-1)/2,(k-1)/2}$ (that is, two stars with $(k-1)/2$ leaves with their centres joined by an edge) for $k \ge 7$ odd, even though almost $1/3$ of vertices of $K_{(k-1)/2,k}$ have degree at least k and the size of the complement of a maximal independent set in $B_{(k-1)/2,(k-1)/2}$ is 2, i.e., its relative size with respect to the whole bistar is very small, in particular at most $1/4$.

We prove that Conjecture 1.1 is asymptotically correct for dense graphs.

Theorem 1.2 *Let $0 < r \leq 1/2$ and $q > 0$. Then for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $k \geq qn$, any graph of order n with at least rn vertices of degree at least $(1+\varepsilon)k$ contains every tree of order at most k with colour classes V_1, V_2 such that $|V_1| \leq rk$.*

This extends the main result of [10], which is a special case of Theorem 1.2 for $r = 1/2$. While we use and extend some of their techniques, our analysis is more complex.

2 Sketch of the proof

Let G be an inclusion-wise minimal graph satisfying the assumptions of Theorem 1.2. Then the set of vertices of degree less than $(1+\varepsilon)k$ is independent. We may assume that at least $(1+\varepsilon)rn$ vertices have degree at least $(1+\varepsilon)k$ (by erasing a tiny fraction of vertices of degree less than $(1+\varepsilon)k$ we obtain a graph satisfying the above just with a smaller ε than in Theorem 1.2). Let T be a tree of order at most k with a colour class of size at most rk.

We follow a strategy based on the Regularity Method that is commonly employed for embedding trees (e.g. [8,10]). In this approach, the goal always is to find a matching structure in the cluster graph of G with favourable properties. To embed T we shall then fill-up regular pairs of this matching structure (or of clusters "adjacent" to this matching structure) with parts of T. While we cannot go into details of the embedding here due to space constraints, below we define the sets which form the building bricks of our matching structure which allow successful embedding of T. Let us now give details how the matching structure is obtained. This is the heart of our proof and the combinatorial argument to this end is considerably more involved than in the case of the ordinary Loebl-Komlós-Sós conjecture [10].

We apply the regularity lemma [11] on G. Erase all edges within clusters, in irregular pairs, or in regular pairs of very small density. Slightly abusing notation, we still call this graph G. We choose a suitable $\varepsilon' > 0$ depending on ε and call a cluster an L-cluster if the average degree of its vertices is at least $(1+\varepsilon')k$ and otherwise an S-cluster. Subdivide L-clusters and S-clusters into few subclusters in a way that the size of the L-subclusters is approximately $(1-r)/r$ times smaller than the size of the S-subclusters. Define a cluster graph $\mathbf{H}$ with its vertex set being the set of the (sub)clusters of G and with edges between vertices corresponding to (sub)clusters forming a regular pair of substantial density. For $v \in V(\mathbf{H})$, let $\bar{\deg}(v)$ denote the average degree of the vertices in the cluster corresponding to v. Let $\mathbf{L}$ be the set of vertices v

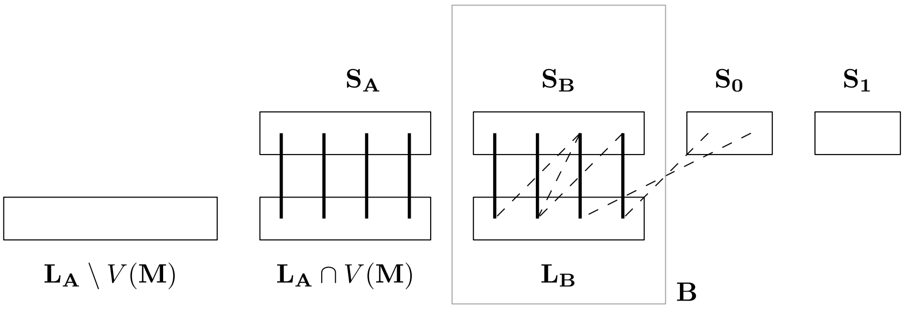

Fig. 2. The structure of the cluster graph $\mathbf{H}$ and the matching $\mathbf{M}$.

with $\bar{\deg}(v) \geq (1+\varepsilon')k$, and let $\mathbf{S}$ be the remaining vertices of $\mathbf{H}$. Then $|\mathbf{L}|$ is slightly larger than $|\mathbf{S}|$.

We consider a matching $\mathbf{M}$ between $\mathbf{L}$ and $\mathbf{S}$ that minimizes the number of vertices $v \in \mathbf{S} \setminus V(\mathbf{M})$ with $\bar{\deg}(v) < (1+\varepsilon')rk$. Let $\mathbf{S_0}$ denote the set of such vertices and let $\mathbf{S_1} = \mathbf{S} \setminus (V(\mathbf{M}) \cup \mathbf{S_0})$ and $\mathbf{S_M} = \mathbf{S} \cap V(\mathbf{M})$. Define $\mathbf{B}$ as the set of clusters $v \in V(\mathbf{M})$ for which there is an alternating path $v_1, v_2, \ldots, v_\ell$ with $v_1 \in \mathbf{S_0}$, $v_\ell = v$, $v_{2i} \in \mathbf{L}$, $v_{2i+1} \in \mathbf{S}$, and $v_{2i}v_{2i+1} \in E(\mathbf{M})$ (see Figure 2). Set $\mathbf{L_B} = \mathbf{L} \cap \mathbf{B}$, $\mathbf{L_A} = \mathbf{L} \setminus \mathbf{L_B}$, $\mathbf{S_B} = \mathbf{S} \cap \mathbf{B}$, and $\mathbf{S_A} = \mathbf{S_M} \setminus \mathbf{S_B}$. Observe that $|\mathbf{S_B}| = |\mathbf{L_B}|$. Therefore, as $|\mathbf{L}|$ is slightly larger than $|\mathbf{S}|$, we deduce that $\mathbf{L_A}$ is non-empty. Also observe that from the definition of $\mathbf{B}$ it follows that $e_{\mathbf{H}}(\mathbf{L_A}, \mathbf{S_0} \cup \mathbf{S_B}) = 0$. We consider two particular subsets of $\mathbf{L}$:

$$\mathbf{L}^+ = \{v \in \mathbf{L} \,:\, \bar{\deg}(v, \mathbf{S_M} \cup \mathbf{S_1}) \geq (1+\varepsilon')(1-r)k\} \,,$$
$$\mathbf{L}^* = \{v \in \mathbf{L} : \bar{\deg}(v, \mathbf{L}) \geq (1+\varepsilon')rk\} \,.$$

Observe that we have $\mathbf{L_A} \subseteq \mathbf{L}^* \cup \mathbf{L}^+$. Finally, set $\mathbf{N} = N(\mathbf{L}^* \cap \mathbf{L_A}) \cap \mathbf{L}$. We are now able to obtain the desired matching structure advertised above (the exact structure is too technical to be described here), unless the cluster graph has the following properties:

(i) $e(\mathbf{L_A}, \mathbf{S_1} \cup \mathbf{L}^*) = 0$, hence $\mathbf{N} \subseteq \mathbf{L_B}$,

(ii) $\bar{\deg}(v) < (1+\varepsilon')rk$ for all $v \in \mathbf{S_A}$,

(iii) $\bar{\deg}(v, \mathbf{S_0}) \geq (1+\varepsilon')rk$ for all $v \in \mathbf{N}$.

By contradiction, we show that $\mathbf{H}$ cannot satisfy all three properties.

From (1) we infer that $\bar{\deg}(v, \mathbf{S_A}) \geq (1+\varepsilon')(1-r)k$ for any vertex $v \in \mathbf{L}^+ \cap \mathbf{L_A}$. As the edges between $\mathbf{L}^+ \cap \mathbf{L_A}$ and $\mathbf{S_A}$ are skewed, (2) gives that $|\mathbf{L}^+ \cap \mathbf{L_A}| < |\mathbf{S_A}|$, and hence $\mathbf{L}^* \cap \mathbf{L_A}$ is non-empty.

Recall that vertices in $\mathbf{L}$ correspond to smaller clusters than vertices in $\mathbf{S}$. Using (3) we deduce that $\mathbf{N}$ can be only slightly bigger than $\frac{r}{1-r}|\mathbf{S_0}|$. On the other hand, by estimating the number of edges of the underlying graph G

between the sets corresponding to $\mathbf{L}^+ \cap \mathbf{L_A}$ and $\mathbf{S_A}$ and the number of edges between the sets corresponding to $\mathbf{L}^* \cap \mathbf{L_A}$ and $\mathbf{N}$, we can calculate that $\mathbf{N}$ needs to be substantially bigger than $|\mathbf{L}^* \cap \mathbf{L_A}| + \frac{r}{1-r}|\mathbf{S_0}|$, a contradiction.

3 A bound on the Ramsey number of skew trees

The Ramsey number $R(G_1, \ldots, G_m)$ is the least number such that any complete graph on $R(G_1, \ldots, G_m)$ vertices with its edges coloured with m colours contains a monochromatic copy of G_i in colour i for some $1 \leq i \leq m$. It is not difficult to see that, if true, both the Loebl–Komlós–Sós conjecture and the Erdős–Sós conjecture would imply that for any pair of trees T_1, T_2 on $k+1$ and $l+1$ vertices, respectively, it holds that $R(T_1, T_2) \leq k + l$. This was shown to be asymptotically true in [10] and even finer asymptotic bound was obtained for $T_1 = T_2$ in [8].

Our Conjecture 1.1 generalizes this consequence for trees of given skew. Suppose we have trees $T_1, \ldots, T_m$ such that the size of the i-th tree is k_i+1 and the size of one of its colour class is at most $(k_i + 1)/m$. Assuming the validity of Conjecture 1.1, we deduce $R(T_1, \ldots, T_m) \leq 2 + \sum_{i=1}^{m} (k_i - 1)$. Indeed, for every vertex v there exists a colour i such that v is incident with at least k_i edges of colour i. Moreover, there exists a colour c such that at least $1/m$ of the vertices are incident with at least k_c edges of colour c. Thus, the subgraph formed by the edges of colour c satisfies the conditions of Conjecture 1.1. Using Theorem 1.2, we prove this consequence to be asymptotically true.

Corollary 3.1 *For trees $T_1, \ldots, T_m$ with $|T_i| = k_i$ and such that one color class of T_i has size at most k_i/m for $1 \leq i \leq m$ we have*

$$R(T_1, \ldots, T_m) \leq \sum_{i=1}^{m} k_i + o\left(\sum_{i=1}^{m} k_i\right).$$

This generalises the asymptotic bound from [10] and can be shown in a very similar manner.

Note that, if true, the Erdős–Sós conjecture would imply the same bound but without the additional restriction on the skew of the trees.

4 Conclusion

We believe that, similarly as in [2,9], one could use the stability method to prove that Conjecture 1.1 holds for large dense graphs. Also methods developed in [1,4,5,6,7] could be applied to prove the (asymptotic) version of Conjecture 1.1 for large values of k even when the host graph is sparse.

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